1 (a) $y' = e^{2x} + \sin(x)$

(i) order 1 (since y' is largest derivative)

(ii) linear since of the form

$$\boxed{1} y' + \boxed{0} \cdot y = \boxed{e^{2x} + \sin(x)}$$

coefficients are #'s and x's

1 (b) $y'' + \sin(2x) y' + 2y = e^x$

(i) order 2 (since y'' is largest derivative)

(ii) linear since of the form

$$\boxed{1} y'' + \boxed{\sin(2x)} y' + \boxed{2} y = \boxed{e^x}$$

coefficients are #'s and x's

$\boxed{①(c)}$ $y''' + 3y'' + 4y' + 12y = x^2 + x - 1$

(i) order 3 (since y''' is largest derivative)

(ii) linear since of the form

$$\boxed{1} \cdot y''' + \boxed{3} \cdot y'' + \boxed{4} \cdot y' + \boxed{12} \cdot y = \boxed{x^2 + x - 1}$$

coefficients are #'s and x's

---

①(d) $2y'' + yx^3y' + x^2y = 10$

(i) order 2 (since y'' is largest derivative)

(ii) not linear

$$\boxed{2} y'' + \boxed{yx^3} y' + \boxed{x^2} y = \boxed{10}$$

these are #'s and x's

but this coefficient of y' has a y in it, so not linear

① (e)

$$x^2 \frac{d^{10}y}{dx^{10}} - 5\frac{d^2y}{dx^2} + \sin(x)\frac{dy}{dx} - 2\sin(x)y = e^x$$

(ii) order 10 (since $\frac{d^{10}y}{dx^{10}}$ is largest derivative)

(iii) linear since of the form

$$\boxed{x^2} \frac{d^{10}y}{dx^{10}} \boxed{-5}\frac{d^2y}{dx^2} + \boxed{\sin(x)}\frac{dy}{dx} \boxed{-2\sin(x)}y = \boxed{e^x}$$

coefficients are #'s and x's

---

① (f) $(y^2+1)\frac{dy}{dx} + e^y y = 2x$

(i) order 1 (since $\frac{dy}{dx}$ is largest derivative)

(ii) not linear

$$\boxed{(y^2+1)}\frac{dy}{dx} + \boxed{e^y}y = \boxed{2x}$$

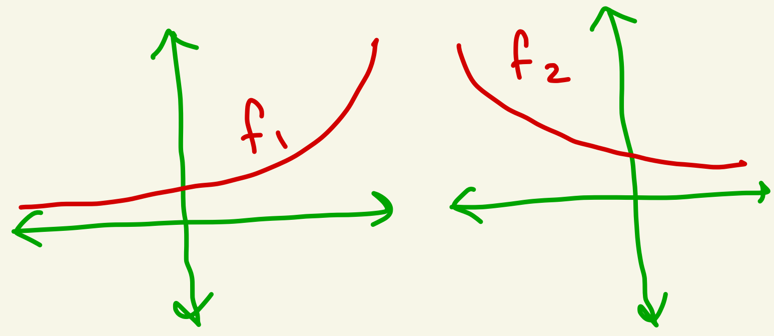
← only #'s and x's

but these coefficients have y's hence not linear

2 (a) First note that $f_1(x) = e^{2x}$ and $f_2(x) = e^{-2x}$ are both defined on $I = (-\infty, \infty)$.

We have

$$\left.\begin{aligned} f_1(x) &= e^{2x} \\ f_1'(x) &= 2e^{2x} \\ f_1''(x) &= 4e^{2x} \end{aligned}\right\} \text{all defined on } I=(-\infty,\infty)$$



Thus,

$$f_1'' - 4f_1 = 4e^{2x} - 4e^{2x} = 0$$

So, f_1 satisfies $y'' - 4y = 0$

Also,

$$\left.\begin{aligned} f_2(x) &= e^{-2x} \\ f_2'(x) &= -2e^{-2x} \\ f_2''(x) &= 4e^{-2x} \end{aligned}\right\} \text{all defined on } I=(-\infty,\infty)$$

Thus,

$$f_2'' - 4f_2 = 4e^{-2x} - 4e^{-2x} = 0$$

So, f_2 satisfies $y'' - 4y = 0$.

2(b)

Let $f(x) = c_1 f_1(x) + c_2 f_2(x) = c_1 e^{2x} + c_2 e^{-2x}$.

Then,

$$f'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x}$$

$$f''(x) = 4c_1 e^{2x} + 4c_2 e^{-2x}$$

Thus,

$$f'' - 4f = 4c_1 e^{2x} + 4c_2 e^{-2x} - 4\left(c_1 e^{2x} + c_2 e^{-2x}\right) = 0$$

So, f satisfies $y'' - 4y = 0$

2(c) We know from part (b) that $f(x) = c_1 e^{2x} + c_2 e^{-2x}$ satisfies $y'' - 4y = 0$.

Now we need it to also satisfy $y'(0) = 0$, $y(0) = 1$.

Note that

$$f'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x}$$

So, we need

$$0 = f'(0) = 2c_1 e^{2(0)} - 2c_2 e^{-2(0)} = 2c_1 - 2c_2$$

$$1 = f(0) = c_1 e^{2(0)} + c_2 e^{2(0)} = c_1 + c_2$$

That is we need to solve

$$\boxed{\begin{aligned} 2c_1 - 2c_2 &= 0 \quad (1) \\ c_1 + c_2 &= 1 \quad (2) \end{aligned}}$$

which is equivalent to

$$\boxed{\begin{aligned} c_1 - c_2 &= 0 \quad (1) \\ c_1 + c_2 &= 1 \quad (2) \end{aligned}}$$

(1) gives $c_1 = c_2$.

Plug this into (2) to get $c_2 + c_2 = 1$.

So, $c_2 = \frac{1}{2}$.

Thus, $c_1 = c_2 = \frac{1}{2}$.

Thus,

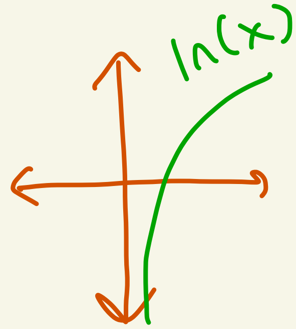
$$f(x) = \frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x}$$

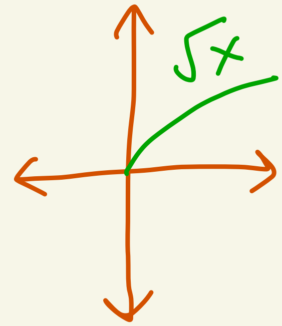
satisfies

$$y'' - 4y = 0, \quad y'(0) = 0, \quad y(0) = 1$$

3 Let $\varphi(x) = 2\sqrt{x} - \sqrt{x}\ln(x)$

Note that φ is defined for all $x > 0$ that is on $I = (0, \infty)$.





3(a) We have

$$\varphi(x) = 2x^{1/2} - x^{1/2} \cdot \ln(x)$$

$$\varphi'(x) = x^{-1/2} - \frac{1}{2}x^{-1/2} \cdot \ln(x) - x^{1/2} \cdot \frac{1}{x}$$

$$= x^{-1/2} - \frac{1}{2}x^{-1/2} \cdot \ln(x) - x^{-1/2}$$

$$= -\frac{1}{2}x^{-1/2} \cdot \ln(x)$$

$$\varphi''(x) = \frac{1}{4}x^{-3/2} \cdot \ln(x) - \frac{1}{2}x^{-1/2} \cdot \frac{1}{x}$$

$$= -\frac{1}{2}x^{-3/2} + \frac{1}{4}x^{-3/2} \cdot \ln(x)$$

all defined on $I = (0, \infty)$

Thus,

$$4x^2 \cdot \varphi'' + \varphi = 4x^2\left[-\frac{1}{2}x^{-3/2} + \frac{1}{4}x^{-3/2} \cdot \ln(x)\right] + 2x^{1/2} - x^{1/2} \cdot \ln(x)$$

$$= -2x^{1/2} + x^{1/2} \cdot \ln(x) + 2x^{1/2} - x^{1/2} \cdot \ln(x) = 0$$

Therefore, $\varphi(x) = 2\sqrt{x} - \sqrt{x}\ln(x)$

satisfies $4x^2y'' + y = 0$ on $I = (0, \infty)$.

Now let's verify that $\varphi(x)$ satisfies

$y'(1) = 0$ and $y(1) = 2$.

We have that

$$\varphi(1) = 2\sqrt{1} - \sqrt{1} \cdot \underbrace{\ln(1)}_{0} = 2$$

$$\varphi'(1) = -\frac{1}{2}(1)^{-1/2} \cdot \underbrace{\ln(1)}_{0} = 0$$

Thus, from (a) and the above we know that

φ solves the initial-value problem

$$4x^2y'' + y = 0, \quad y'(1) = 0, \quad y(1) = 2$$

④ Let $f(x) = \cos(2x)$

Then

$$\left.\begin{aligned} f(x) &= \cos(2x) \\ f'(x) &= -2\sin(2x) \\ f''(x) &= -4\cos(2x) \\ f'''(x) &= 8\sin(2x) \end{aligned}\right\} \text{all defined on } I = (-\infty, \infty)$$

And,

$$f''' + 3f'' + 4f' + 12f$$

$$= 8\sin(2x) + 3(-4\cos(2x)) + 4(-2\sin(2x)) + 12\cos(2x)$$

$$= 8\sin(2x) - 12\cos(2x) - 8\sin(2x) + 12\cos(2x)$$

$$= 0$$

Thus, $f(x) = \cos(2x)$ satisfies

$$y''' + 3y'' + 4y' + 12y = 0$$

Now let's verify that f satisfies the conditions $y''(0)=-4,\ y'(0)=0,\ y(0)=1$

We have

$$f''(0) = -4\cos(2\cdot 0) = -4(1) = -4$$

$$f'(0) = -2\sin(2\cdot 0) = -2(0) = 0$$

$$f(0) = \cos(2\cdot 0) = 1$$

It does!

Therefore, $f(x)=\cos(2x)$ solves

$$y''' + 3y'' + 4y' + 12y = 0$$
$$y''(0) = -4,\ y'(0) = 0,\ y(0) = 1$$